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HOMOGENEOUS POLYNOMIAL IDENTITIES

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ABSTRACT
PI-algebras are studied by attaching invariants to the homogeneous identities analogous to the invariants of the multilinear identities studied by Regev. Also, it is shown that every finitely generated PI-algebra is polynomially bounded.

1. Introduction

This paper studies the sets of polynomials which occur as identities for a given algebra. Let A be a PI-algebra defined over the field of characteristic zero, F; let $F\langle x_1, x_2, \cdots\rangle = F\langle X\rangle$ be the free associative algebra over F in a countable set X; let $Q \subseteq F\langle X\rangle$ be the set of polynomials which vanish on A; and let $V_n \subseteq F\langle x\rangle$ be the set of multilinear, homogeneous polynomials of degree n in $x_1, \cdots, x_n$. Regev has defined two sets of invariants of $Q : c_n(A)$ and $\chi_n(A)$. $c_n(A)$ = the dimension of the F-vector space $V_n / V_n \cap Q$ and $\chi_n(A)$ = the character of $V_n / V_n \cap Q$ considered as an FS_n-module. The study of these invariants has yielded a number of results; for an account of them, see [1].

Here this technique is modified slightly. Instead of considering V_n, multilinear polynomials in n variables, we consider W_n which will consist of homogeneous polynomials of degree n in a fixed finite set of variables. Working by analogy to the multilinear invariants, we describe the dimension of $W_n / W_n \cap Q$ and its character as a GL(k)-module. We also investigate the codimensions of Q in another family of spaces of homogeneous polynomials. If $(a) = (a_1, a_2, \cdots, a_k)$, set $W_{(a)} = \{f(x_1, \cdots, x_k) \in F\langle x_1, \cdots, x_k\rangle \mid$ each monomial of f has degree a_i in x_i, $i = 1, 2, \cdots, k\}$, then set $c(a) = \dim W_{(a)}/W_{(a)} \cap Q$. One motivation for studying these spaces is that

$$Q \cap F\langle x_1, \cdots, x_k\rangle = \sum_{n=0}^{\infty} \bigoplus (W_n \cap Q) = \sum_{a_1, \cdots, a_k = 0}^{\infty} \bigoplus (W_{(a)} \cap Q).$$

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The results are as follows:

dim $W_n / W_n \cap Q$ is bounded by a polynomial as a function of n (Corollary 4.12).

The S_n-characters of $V_n / V_n \cap Q$ determine the $\mathrm{GL}(k)$-characters of $W_n / W_n \cap Q$ for each k, and conversely, provided $k \geqq n$ (Theorem 2.7).

The $\mathrm{GL}(k)$-characters of $W_n / W_n \cap Q$ determine the $c(a)$, and conversely (Theorem 3.3, Corollary 3.4).

These results all have applications. Corollary 4.12 is used to prove that every finitely generated PI-algebra has polynomially bounded rate of growth (4.13). The proof of this result in Section 4 is self-contained. Corollary 3.4 is used along with a result of Formanek, Halpin, and Li to calculate the height two cocharacter for Q equal to the identities for 2×2 matrices. Theorem 2.7 is used to show that the set of characters $\chi_{S_n}(V_n / V_n \cap Q)$ is closed under the Kronecker product (2.8).

Finally, in Section 5, using a theorem of Procesi we show that, if Q = the identities for $r \times r$ matrices, then dim $W_n / W_n \cap Q$ is bounded above and below by polynomials in n of degree $r^2(k-1)$.

It gives me great pleasure to thank two people who were instrumental in bringing about this paper: my thesis advisor Professor Herstein who gave me support, guidance, and encouragement; also, Professor Regev for many helpful conversations and suggestions. In particular, it was on Professor Regev's advice that I considered $W_n / W_n \cap Q$ and on the basis of his belief that I attempted to prove Corollary 4.12. Thanks are also due to the referee for a number of useful suggestions including the remark following Corollary 3.4.

2. Relationship between S_n-cocharacters and $\mathrm{GL}(k)$-cocharacters of Q

We will use some standard notations and identifications. First, V_n the set of multilinear, homogeneous degree n polynomials in $x_1, \cdots, x_n$ is identified with FS_n via

$$\sigma \in S_n \equiv x_{\sigma_1} \cdots x_{\sigma_n} \in V_n.$$

In particular, V_n may be regarded as an $S_n - S_n$ bimodule. FS_n acts on V_n on the right by substitution and on the left by place permutation:

$$\sigma f(x_1, \cdots, x_n) = f(x_{\sigma_1}, \cdots, x_{\sigma_n}), \qquad f \in V_n, \quad \sigma \in S_n;$$

$$y_1 \cdots y_n \sigma = y_{\sigma_1} \cdots y_{\sigma_n}, \qquad y_1 \cdots y_n \text{ a monomial of } V_n, \quad \sigma \in S_n.$$

Next, let V be an F-vector space with basis $\{x_1, \cdots, x_k\}$. Then W_n, or simply W, the set of all homogeneous degree n polynomials in $x_1, \cdots, x_k$ may be identified with $V^{\otimes n}$. The space W has a well known $\mathrm{GL}(k)$-S_n structure. In terms of polynomials, the action $W \otimes FS_n \to W$ may be regarded as an action of FS_n on W by place permutation or, identifying FS_n with V_n, as W substituting into V_n:

$$y_1 \cdots y_n \sigma = y_{\sigma_1} \cdots y_{\sigma_n}, \qquad y_1 \cdots y_n \text{ a monomial of } W, \quad \sigma \in S_n\,;$$

$$y_1 \cdots y_n f(x_1, \cdots, x_k) = f(y_1, \cdots, y_n), \qquad y_1 \cdots y_n \text{ a monomial of } W, \quad f \in V_n.$$

We will assume that the reader is familiar with the representation theory of S_n and $\mathrm{GL}(k)$ as they relate to V_n and W. The facts may be found in [3] or [7].

The span of $\mathrm{GL}(k)$ in $\mathrm{End}_F(W, W)$, which is the same as the centralizer of FS_n in $\mathrm{End}_F(W, W)$, will be written $B(n, k)$ or simply B. The set of partitions of n is denoted $\mathrm{Par}(n)$, and for $\lambda \in \mathrm{Par}(n)$ the corresponding S_n-character is denoted χ_λ. The set of partitions of n of height $\leqq k$ is denoted $\Lambda_k(n)$ and for $\lambda \in \Lambda_k(n)$ the corresponding $\mathrm{GL}(n)$-character is denoted ϕ_λ.

Since Q is closed under substitution, the following is trivial:

Lemma 2.1. (a) $FS_n(Q \cap V_n) \subseteq Q \cap V_n$,
(b) $W(Q \cap V_n) \subseteq Q \cap W$,
(c) $\mathrm{GL}(k)(Q \cap W) \subseteq Q \cap W$.

If $w = y_1 \cdots y_n \in W$ is a monomial, let $R = R(w) = \{\sigma \in S_n \mid w\sigma = w\}$ and let $s = s(w) = \Sigma_R\, \sigma \in FS_n$. Lemma 2.2 is an important technicality. It is really a theorem about specialization and linearization.

Lemma 2.2. $Bw = Ws$.

Proof. First

$$ws = w \sum_R \sigma = \sum_R w\sigma = |R|\, w \tag{2.2.1}$$

by definition of R. So,

$$w = \frac{1}{|R|}\, ws \in Ws.$$

Since $1 \in B$, $Bw \subseteq Bws \subseteq Ws$.

Now, assume $Bw \neq Ws$. Since W is completely reducible as a B-module, it can be decomposed into a direct sum of submodules $W = Ws \oplus W_1 = Bw \oplus W_1 \oplus W_2$. Let $\pi : W \to W_2$ be the projection map. $\pi \in \mathrm{End}_B(W, W)$, so π can be realized by some $a \in FS_n$, i.e., $\pi(v) = va$ for all $v \in W$. Let T be a left transversal for R in S_n and let $a = \Sigma\, \alpha_\sigma \sigma$. Then

$$0=\pi(w)=wa=\sum \alpha_\sigma w\sigma=\sum_{\tau\in T}\sum_{\rho\in R}\alpha_{\rho\tau}w\rho\tau$$

$$=\sum_{\tau\in T}\sum_{\rho\in R}\alpha_{\rho\tau}w\tau=\sum_{\tau\in T}\Big(\sum_{\rho\in R}\alpha_{\rho\tau}\Big)w\tau.$$

The $w\tau$ are linearly independent, so

$$\sum_{\sigma\in R\tau}\alpha_\sigma=0 \qquad \text{for each } \tau. \tag{2.2.2}$$

Finally, we calculate $\pi(Ws)$:

$$\pi(Ws)=Wsa=W\sum_{S_n}\alpha_\sigma sa=W\Big(\sum_{\tau\in T}\sum_{\rho\in R}\alpha_{\rho\tau}s\rho\tau\Big)=W\Big(\sum_{\tau\in T}\Big(\sum_{\rho\in R}\alpha_{\rho\tau}\Big)s\tau\Big)$$

which is zero by (2.2.2). But $\pi(W_2)\neq 0$ gives a contradiction.

COROLLARY 2.3. *Let $a\in V_n$, $w\in W$ a monomial, and $s=s(w)$. Then $wa\in Q\cap W$ if and only if $sa\in Q\cap V_n$.*

PROOF. If $sa\in Q$, $wsa\in Q$ by 2.1b. But $ws=|R|w$ by 2.2.1, so $|R|wa\in Q$, and $wa\in Q$. Conversely, if $wa\in Q$, choose V with dimension n. By Theorem 2.2, $B(n,n)w=Ws$, and this s is independent of k. So, there is a $b\in B(n,n)$, $bw=(x_1\cdots x_n)s$ and $x_1\cdots x_ns=$ image of s under the identity substitution which equals s. By 2.1c, $bwa\in Q$, so $sa\in Q$.

COROLLARY 2.4. *With notation as in 2.3, $wa=0$ if and only if $sa=0$.*

PROOF. This is Corollary 2.3 in the case of $Q=0$.

Remark 2.5 is a well known fact. It says that, over an infinite field, T-ideals are homogeneous.

REMARK 2.5. Let w_i be a set of monomials in W with distinct multi-degrees, i.e., $w_iFS_n\cap w_jFS_n=0$ if $i\neq j$. If $a_i\in V_n$ and $\Sigma\, w_ia_i\in Q$, then $w_ia_i\in Q$ for each i.

LEMMA 2.6. *If $V_n=(V_n\cap Q)\oplus I_n$ is a direct sum of FS_n-modules, then as $\mathrm{GL}(k)$-modules $W=(W\cap Q)\oplus WI_n$.*

PROOF. Since $1\in V_n$, $W=W(V_n\cap Q)+WI_n$. By 2.1b, $W(V_n\cap Q)\subseteq W\cap Q$. Next we claim that $WI_n\cap Q=0$. Assume not. Since $FS_nI_n=I_n$, there is a sum $\Sigma\, w_ia_i\in Q$, $a_i\in I_i$, w_i as in 2.5. By 2.5, there is a $0\neq w_ia_i\in Q$. However, by Corollaries 2.3 and 2.4, $0\neq sa_i\in Q$ for appropriate $s\in FS_n$. But, $sa_i\in Q\cap I_n=0$, a contradiction.

Now, $W(V_n \cap Q) \subseteq (W \cap Q)$ has trivial intersection with WI_n, so $W(V_n \cap Q) \oplus WI_n = W$ is a direct sum decomposition. Finally, $W = W(V_n \cap Q) \oplus WI_n \subseteq (W_n \cap Q) \oplus WI_n \subseteq W$.

Theorem 2.7 is the main result of this section.

THEOREM 2.7. *If*

$$\chi_{S_n}(V_n / W_n \cap Q) = \sum_{\lambda \in \mathrm{Par}(n)} m_\lambda \chi_\lambda$$

and if

$$\chi_{\mathrm{GL}(k)}(W / W \cap Q) = \sum_{\lambda \in \Lambda_k(n)} m'_\lambda \phi_\lambda$$

then $m'_\lambda = m_\lambda$ *for each* $\lambda \in \Lambda_k(n)$.

PROOF. Since FS_n is semisimple, V_n can be written $V_n = (V_n \cap Q) \oplus \Sigma_i FS_n e_i$, where the e_i are minimal idempotents on standard tableaux, and the number of tableaux of shape λ is m_λ, for each λ. By Lemma 2.6,

$$W = (W \cap Q) \oplus \sum WFS_n e_i = (W \cap Q) \oplus \sum We_i$$

the sum being over those e_i on diagrams of height $\leqq k$ because of the FS_n-structure of W. So $\chi_{\mathrm{GL}(k)}(W/W \cap Q) = \chi_{\mathrm{GL}(k)}(\bigoplus We_i) = \Sigma_{\lambda \in \Lambda_k(n)} m_\lambda \phi_\lambda$.

REMARK. The S_n-character of $V_n / V_n \cap Q$ always determines the $\mathrm{GL}(k)$-character of $W_n / W_n \cap Q$. The $\mathrm{GL}(k)$-character of $W_n / W_n \cap Q$ determines the S_n-character of $V_n / V_n \cap Q$ only when $k \geqq n$ and otherwise gives only partial information.

COROLLARY 2.8. $\chi_n(A) \hat{\otimes} \chi_m(A) \geqq \chi_{n+m}(A)$, *the inequality being componentwise.*

PROOF. In the construction of $W_n = V^{\otimes n}$ choose $\dim V = k \geqq n + m$, and let $W_n = J_n \oplus (Q \cap W_n)$, $W_m = J_m \oplus (Q \cap W_m)$, and $W_{m+n} = J_{n+m} \oplus (W \cap Q_{n+m})$.

$$W_{n+m} = W_n \otimes W_m = (J_n \otimes J_m) \oplus (J_n \otimes (Q \cap W_m))$$

$$\oplus ((Q \cap W_n) \otimes J_m) \oplus ((Q \cap W_m) \otimes (Q \cap W_n)).$$

The operation $\otimes$ in W corresponds to multiplication of polynomials, so each of the last three summands is contained in $Q \cap W_{n+m}$. Therefore, there is an injection $J_{n+m} \to J_n \otimes J_m$, and the corollary follows from Theorem 2.7 and the relationship between $\otimes$ on $\mathrm{GL}(k)$-characters and $\hat{\otimes}$ on S_n-characters.

COROLLARY 2.9. $\text{height}(\chi_n(A)) + \text{height}(\chi_m(A)) \geqq \text{height}(\chi_{n+m}(A))$.

PROOF. Follows from the Littlewood–Richardson rule and Corollary 2.8.

3. Cocharacters and Poincaré series

With notation as in Section 2, there is another set of invariants which may be attached to the set of identities Q. Since $Q \cap F\langle x_1, \cdots, x_k \rangle$ is a homogeneous ideal of $F\langle x_1, \cdots, x_k \rangle$, the quotient is multigraded by degree. If $c(a)$ = the dimension of the part of $F\langle x_1, \cdots, x_k \rangle / Q \cap F\langle x_1, \cdots, x_k \rangle$ with degree $(a) = (a_1, \cdots, a_k)$, then the formal power series $\sum_{(a)=(a_1,\cdots,a_k)} c(a) t_1^{a_1} \cdots t_k^{a_k}$ is called the Poincaré series for $F\langle x_1, \cdots, x_k \rangle / F\langle x_1, \cdots, x_k \rangle \cap Q$, and the numbers $c(a)$ are called the Poincaré coefficients. More formally, if $w \in W_n$ is a monomial with degree $(a_1, \cdots, a_k)$, then $c((a_1, \cdots, a_k)) = \dim wFS_n / wFS_n \cap Q$. In this section we will show that the Poincaré coefficients $c(a)$ determine and are determined by the multiplicities in the cocharacter m_λ, as defined in Theorem 2.7.

One trivial observation which will be needed is that the numbers $c(a)$ depend only on the numbers $a_1, \cdots, a_k$ and not on their order. In particular, once $c(a)$ is determined for all $(a) = (a_1, \cdots, a_k)$ with $a_1 \geqq a_2 \geqq \cdots \geqq a_k$ it is known for all (a). So, we will concentrate on $c(\lambda)$, $\lambda \in \text{Par}(n)$.

LEMMA 3.1. *If $A, B \subseteq V_n$ are S_n-modules, and if $A \cap B = 0$, and w is a monomial of W, then $wA \cap wB = 0$.*

PROOF. Assume not. Then we can choose $0 \neq wa = wb$ for some $a \in A$, $b \in B$. Let $s = s(w)$. By Corollary 2.4, $sa \neq 0$, and since $w(a-b) = 0$, $s(a-b) = 0$ and $sa = sb$. But then $sa \in A \cap B = 0$, a contradiction.

One more preliminary is needed. It can be found in ([7], 26.3(i)).

LEMMA 3.2. *Let $e \in FS_n$ be a minimal idempotent gotten from a tableau of shape λ, and let $w \in W$ be a monomial with degree $(a_1, \cdots, a_k)$. Then $\dim_F(wFs_n e)$ = the number of semistandard tableaux of shape λ and type $(a_1, \cdots, a_k)$.*

Now, for each λ, $\mu \in \text{Par}(n)$, let $\alpha_{\lambda\mu}$ = the number of semistandard tableaux of type λ and shape μ, sometimes called the Kostka numbers. Consider $\bar{c} = c(\lambda)$ and $\bar{m} = m(\mu)$ as vectors in a $|\text{Par}(n)|$-dimensional vector space, and $T = (\alpha_{\lambda\mu})_{\lambda,\mu\in\text{Par}(n)}$ as a linear transformation on that space.

THEOREM 3.3. *$\bar{c} = T\bar{m}$, i.e., for all $\lambda \in \text{Par}(n)$ $c(\lambda) = \sum_{\mu\in\text{Par}(n)} \alpha_{\lambda\mu} m_\mu$.*

The matrix T is invertible and the inverse can be computed explicitly (see [7]). Let $S = (\beta_{\lambda\mu})$ be the inverse matrix.

COROLLARY 3.4. $\bar{m} = S\bar{c}$, *i.e.*, $c(\lambda) = \sum_{\mu \in \mathrm{Par}(n)} \beta_{\lambda\mu} m_\mu$.

PROOF OF 3.3. As in the proof of Theorem 2.7, write $V_n = Q_n \oplus I_n$ as FS_n-modules, $Q_n = Q \cap V_n$ and $I = \oplus FS_n e_i$. Then $wV_n = W_{(a)} = wQ_n \oplus wI_n$ by Lemma 3.1. Claim: $wQ_n = W_{(a)} \cap Q$. By 2.1b, $wQ_n \subseteq W_{(a)} \cap Q$. If the containment were proper, there would be a polynomial $0 \neq f \in wI \cap Q$. Write $f = wi \in Q$, $i \in I$. By Lemma 2.3, $s(w)i \in Q \cap I$, and $si \neq 0$ by Lemma 2.4, a contradiction. So $W_{(a)} = (W_{(a)} \cap Q) \oplus wI_n$, and to complete the proof it suffices to calculate $\dim wI_n$. By Lemma 3.1, $wI_n = \oplus wFS_n e_i$, so by Lemma 3.2, $\dim wI_n = \sum_{\mu \in \mathrm{Par}(n)} \alpha_{\lambda\mu} m_\mu$.

REMARK. It is easy to see that the GL(k)-cocharacters and Poincaré series must be related. Let

$$A = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_k \end{pmatrix}, \qquad t_1, \cdots, t_k \in F$$

be a diagonal matrix. Then the trace of A acting on $W_n / W_n \cap Q$ is $\sum_{a_1 + \cdots + a_n = n} c(a) t_1^{a_1} \cdots t_k^{a_k}$, so in some sense, the Poincaré series is the GL(k)-character of $\bigoplus W_n / W_n \cap Q$.

An important Poincaré series is calculated in [4]. Corollary 3.4 can be used to translate that result into a statement about cocharacters. Let $Q =$ the identities for 2×2 matrices and let $k = 2$. Then Formanek, Halpin, and Li [4] show that as formal power series

$$\text{(FHL)} \qquad \sum_{i,j=0} c(i,j) t^i s^j = (1-s)^{-1}(1-t)^{-1} + st(1-s)^{-2}(1-t)^{-2}(1-st)^{-1}.$$

THEOREM 3.5. *If* $Q =$ *the identities of* F_2, *then* $m_{(p,0)} = 1$ *and* $m_{(p,q)} = (p-q+1)q$ *for* $q > 0$.

PROOF. Expanding the right-hand side of (FHL) in a power series and equating coefficients gives that for $i \geqq j$, $c(i,j) = 1 + \sum_{k=0}^{j} (i-k)(j-k)$. The determinantal form [7] and 3.4 imply that $m_{(p,0)} = c(p,0)$ and $m_{(p,q)} = c(p,q) - c(p+1, q-1)$ if $q > 0$. So $m_{(p,0)} = 1$ and if $q > 0$,

$$m_{(p,q)} = \left(1 + \sum_{k=0}^{q} (p-k)(q-k)\right) - \left(1 + \sum_{k=0}^{q-1} (p+1-k)(q-1-k)\right).$$

Shifting indices in the second expression and adding now gives the desired result.

4. Polynomial growth

In this section we prove that $\dim(W_n / W_n \cap Q)$ is bounded by a polynomial in n. The proof is combinatorial, and much of the machinery is similar to that in the proof of Shirshov's theorem in [6].

A few more definitions are needed. Let M = the free monoid on $\{x_1, x_2, \cdots\}$, so that $F\langle X\rangle = F[M]$. M is graded by degree, and the words of degree n are denoted by M_n.

Next, put a partial order on M as follows: if $u, v \in M$, $u = x_{i_1} \cdots x_{i_a}$, $v = x_{j_1} \cdots x_{j_b}$ then $u < v$ if there is an $1 \leqq s \leqq a, b$ such that $i_1 = j_1, \cdots, i_{s-1} = j_{s-1}$, and $i_s < j_s$. Note that two words are unrelated precisely when one is an initial segment of the other. So 1 is unrelated to all other words.

LEMMA 4.1. (a) *If $u < v$, then $uw < vq$ for all $w, q \in M$.*

(b) *If $u < v$, then $wu < wv$ for all $w \in M$.*

(c) *If $a > b$ are positive integers and $x_k > u$ then $x_k^b u < x_k^a v$.*

The proof is elementary and we omit it.

DEFINITIONS. $u = u_1 u_2 \cdots u_d \in M$ is called a dominant factorization of u of length d, or a d-dominant factorization of u, if for all $1 \neq \sigma \in S_d$, $u > u_{\sigma 1} \cdots u_{\sigma d}$. If u contains no subword with a dominant factorization of length d, u will be called d-clean. If u contains a subword $u' = u_1 \cdots u_d$ such that $u_1 > u_2 > \cdots > u_d$, u is d-bad, and if u is not d-bad, it is d-good.

For example, $u = x_3 x_2 x_1$ is 3-bad and has a 3-dominant factorization. A more subtle example is $u = x_1 x_2 x_1^2 x_2 x_1$. A 3-dominant factorization of u is given by $u = (x_1 x_2)(x_1^2 x_2)(x_1)$. u is 4-clean, but it is 3-good and 2-bad.

LEMMA 4.2. *If u is d-clean, it is d-good.*

PROOF. Assume u is d-bad. Then u has a subword u' which can be written $u' = u_1 \cdots u_d$, where $u_1 > u_2 > \cdots > u_d$.

CLAIM. $u_1 \cdots u_d$ is a d-dominant factorization. Let $1 \neq \sigma \in S_d$ and let j be as small as possible subject to $\sigma(j) \neq j$. Then $\sigma(j) > j$, otherwise $\sigma(j)$ would be fixed by σ and so by σ^{-1}, which is impossible. $u_{\sigma 1} \cdots u_{\sigma j} = u_1 \cdots u_{j-1} u_{\sigma j}$. By hypothesis $u_{\sigma j} < u_j$, so by 3.1, $u_{\sigma 1} \cdots u_{\sigma j} < u_1 \cdots u_j$, and $u_{\sigma 1} \cdots u_{\sigma d} < u_1 \cdots u_d$.

This next lemma is analogous to Latyshev's lemma (cf. [10] theorem 1.3). Recall that Q contains the proper identity $x_1 \cdots x_d - \Sigma_{\sigma \neq 1} \alpha_\sigma x_{\sigma 1} \cdots x_{\sigma d}$.

LEMMA 4.3. *$W_n / W_n \cap Q$ is spanned by the d-clean monomials of W_n.*

PROOF. First, since all words of W_n have the same degree, the partial order $<$ is a total order, and it is the lexicographical order. Now, assume the lemma is false, and let u be the smallest word of W_n such that u cannot be expressed modulo $Q \cap W_n$ as a linear combination of d-clean monomials. In particular, u itself is not d-clean, so u can be written $u = u_0u_1 \cdots u_du_{d+1}$, where $u_1 \cdots u_d > u_{\sigma 1} \cdots u_{\sigma d}$ for all $1 \neq \sigma \in S_d$, hence $u_0u_1 \cdots u_{d+1} > u_0u_{\sigma 1} \cdots u_{\sigma d}u_{d+1}$. Thus by the minimality of u, each $u_0u_{\sigma 1} \cdots u_{\sigma d}u_{d+1}$ is a linear combination of d-clean monomials. But $u_0 \cdots u_{d+1} = \sum_{\sigma \neq 1} \alpha_\sigma u_0 u_{\alpha 1} \cdots u_{\sigma d}u_{d+1}$, since $u_1 \cdots u_d - \sum_{\sigma \neq 1} \alpha_\sigma u_{\sigma 1} \cdots u_{\sigma d} \in Q$. This gives a contradiction and completes the proof.

If $A_n \subseteq M_n$, $n = 0, 1, 2, \cdots$, denote $A = \{A_n\}_1^\infty$, and call A a graded subset of M. The most important example here is the graded subset of M of d-clean monomials. To study it, a number of preliminaries are needed.

LEMMA 4.4. *Let A, B, and C be graded subsets of M. Assume*

(1) *for every $w \in A$ there exists $w_1 \in B$ and $w_2 \in C$ such that $w = w_1w_2$, and*

(2) *$|B_n|$, $|C_n|$ are each polynomially bounded as functions of n.*

Then $|A_n|$ is bounded by a polynomial in n.

PROOF. For each $w \in A$, choose w_1 and w_2 as in (1). Then A_n has a one-to-one mapping into $\bigcup_{k=0}^n B_k \times C_{n-k}$ given by $w \to (w_1, w_2)$. The map is one-to-one as it has a right inverse $(w_1, w_2) \to w_1w_2$. So

$$|A_n| \leqq \left| \bigcup_{k=0}^{n} B_k \times C_{n-k} \right| \leqq \sum_{k=0}^{n} |B_k \times C_{n-k}| = \sum_{k=0}^{n} |B_k| \cdot |C_{n-k}|.$$

By (2) we may assume $|B_n| \leqq g_1(n)$, $|C_n| \leqq g_2(n)$ for all n, and we may take g_1 and g_2 to be increasing. So,

$$|A_n| \leqq \sum_{k=0}^{n} |B_k|\,|C_{n-k}| \leqq \sum_{k=0}^{n} g_1(k)g_2(n-k) \leqq \sum_{k=0}^{n} g_1(n)g_2(n) = (n+1)g_1(n)g_2(n),$$

which is a polynomial in n.

COROLLARY 4.5. *Let $A, B^{(1)}, \cdots, B^{(t)}$ be graded subsets of M such that*

(1) *for every $w \in A$ there exists $w_1 \in B^{(1)}, \cdots, w_t \in B^{(t)}$ such that $w = w_1w_2 \cdots w_t$, and*

(2) *each $|B_n^{(i)}|$ is polynomially bounded as a function of n.*

Then $|A_n|$ is bounded by a polynomial in n.

PROOF. By induction on t, the case of $t = 2$ is done by Lemma 3.4. Assume that the theorem is true for $t - 1$. Letting C_n in the above lemma be the set of all products $w_2w_3 \cdots w_t$ of degree n, where $w_i \in B^{(i)}$, $i = 2, \cdots, t$, the proof follows in a straightforward manner.

Let $M_n^{k,a}$ be the subset of M_n consisting of words in $\{x_1,\cdots,x_k\}$ in which the first letter is not x_k and in which x_k never occurs with exponent $\geqq a$. Let $M^{k,a}$ be the graded subset $\{M_n^{k,a}\}_{n=1}^{\infty}$. Let $Y\subset M^{k,a}$ be the set $\{x_ix_k^b \mid i=1,2,\cdots,k-1, b=0,1,\cdots,a-1\}$. It is obvious that every word in $M^{k,a}$ can be written in a unique way as a word in the elements of Y, and that $|Y|=a(k-1)$. We order the elements of Y via $x_ix_k^b<x_j\cdot x_k^c$ if either $i<j$ or if $i=j$ and $b<c$. Then the elements of Y can be enumerated $y_1<y_2<\cdots<y_{a(k-1)}$. Define a Y-order on $M^{k,a}$, $\ll$, similar to the original X-order, as follows: $y_{i_1}\cdots y_{i_a}\ll y_{j_1}\cdots y_{j_b}$ if either $i_1<j_1$ or if $i_1=j_1,\cdots,i_{c-1}=j_{c-1}$, $i_c<j_c$, for some c. Clearly, Lemmas 4.1 and 4.2 continue to hold with $\ll$ replacing $<$.

Corresponding to the two orders for the elements of $M^{k,a}$ there are two notions of dominant factorization and of d-clean and d-bad. We now compare the two orders and the corresponding notions. When the terms "dominant factorization" etc. are used without qualification they will mean with respect to $<$.

Lemma 4.6. (a) *If $w,w'\in M_n^{k,a}$ and $w\ll w'$, then $w<w'$.*

(b) *If $w\in M^{k,a}$ is d-clean with respect to $<$, then it is d-clean with respect to $\ll$.*

(c) *If $w\in M^{k,a}$ has a subword with a d-dominant factorization with respect to $\ll$, and if $e>0$, then $x_k^e w$ has a subword with a $(d+1)$-dominant factorization with respect to $<$.*

Remark. If we assume in (a) only that $w,w'\in M^{k,a}$ but not that they have the same degree, we can conclude only that either $w<w'$ or that $w'=wx_k^u w''$, for some $w''\in M^{k,a}$.

Proof. Let $w=y_{i_1}\cdots y_{i_m}$, $w'=y_{j_1}\cdots y_{j_s}$. Assume t is as small as possible such that $i_t\neq j_t$, so $i_t<j_t$. Say $y_{i_t}=x_ix_k^b$ and $y_{j_t}=x_jx_k^c$. Let $w_0=y_{i_1}\cdots y_{i_{t-1}}=y_{j_1}\cdots y_{j_{t-1}}$.

Case 1: $i<j$. Then $x_i<x_j$, so $w_0x_i<w_0x_j$ by 3.1b, so $w<w'$ by 3.1a.

Case 2: $i=j$, $b<c$. First, we claim that $t\neq m$. For, if $t=m$, $\deg=\deg_x n=\deg w=\deg w_0+\deg y_{i_j}=\deg w_0+1+b$. On the other hand, $n=\deg w'\geqq \deg w_0y_{j_t}=\deg w_0+1+c$. So $b\geqq c$, a contradiction. Thus $y_{i_{t+1}}\neq 1$, therefore $x_k^{c-b}>y_{i_{t+1}}$, since x_k dominates all elements of Y, so $x_k^c>x_k^by_{i_{t+1}}$. Likewise $w_0w_jx_k^c>w_0x_ix_k^by_{i_{t+1}}$, and $w'>w$. This proves (a), while (b) is an immediate consequence of (a) and the definitions.

To prove (c), we may assume by part (a) that $w=w_0w_1\cdots w_dw_{d+1}$ such that for all $1\neq\sigma\in S_d$ $w_1\cdots w_d\geqq w_{\sigma1}\cdots w_{\sigma d}$. Let $w'=x_kw_0w_1\cdots w_d$ and consider the factorization $w' = u_1 u_2 \cdots u_{d+1}$ where $u_1 = x_k w_0$, $u_2 = w_1, \cdots, u_{d+1} = w_d$. Since x_k ($<$) dominates all elements of Y, this is clearly a $(d+1)$-dominant factorization.

DEFINITIONS. Let $Z = \{z_1, \cdots, z_t\}$ be a finite totally ordered set, say $z_1 < z_2 < \cdots < z_t$. $M(Z)$ is the free monoid on Z, and $M^{t,a}(Z)$ is defined analogously to $M^{k,a}$. Let $A_n(Z) = A_Z(n, t, d) =$ the set of d-clean words of $M(Z)$ of length n, let $A(Z) = \{A_n(Z)\}_{n=1}^{\infty}$. In order to study the rate of growth of $A_n(Z)$ we need to define two auxiliary subsets of $M(Z)$. Let

$$B_n(Z) = B_Z(n, t, d, a) = \{z_t^e w \in A_n(Z) \mid e > 0,\ w \in M^{t,a}(Z)\} \qquad \text{for } n > 0$$

and set $B_0(Z) = \{1\}$. Let $B'_n(Z) = B'_Z(n, t, d, a) = \{z_t^e w \in A_n(Z) \mid a > e \geqq 0,$ $w \in M^{t,a}(Z)\}$.

We will denote $A_n(X)$ by A_n, $A_X(n, k, d)$ by $A(n, k, d)$, and $A(X)$ by A when there is no danger of ambiguity. Likewise for B and B'. Of course, the cardinalities of these sets are independent of the choice of letters.

LEMMA 4.7. (a) $|A(n, k, 2)|$ *is polynomially bounded in n for all fixed k.*
(b) $|A(n, 1, d)|$ *is polynomially bounded in n for all fixed d.*

PROOF. For part (a) consider $w = x_{i_1} \cdots x_{i_n}$. If some $i_j > i_{j+1}$ then $x_{i_j} x_{i_{j+1}}$ would be a 2-bad subword. So, if $w \in A_X(n, k, 2)$ then $i_1 < i_2 < \cdots < i_n$. The converse is clear. Therefore, $|A(n, k, 2)| =$ the number of non-decreasing sequences of length n in $1, \cdots, k = \binom{k+n-1}{n} < (k+n-1)^{k-1}$ which is polynomial in n. Part (b) is trivial, since $A(n, 1, d) = \{x_1^n\}$ so $|A(n, 1, d)| = 1$.

The main theorem in this section is

THEOREM 4.8. $|A(n, k, d)|$ *is bounded by a polynomial in n for all fixed k and d.*

The proof is by a double induction argument. So assume that
(*) $|A(n, k, d')|$ is polynomially bounded in n for all $2 \leqq d' < d$, and then assume
(**) $|A(n, k', d)|$ is polynomially bounded for all $1 \leqq k' < k$.

LEMMA 4.9. *Under the hypothesis (*) and (**), $|B_n|$ is polynomially bounded in n, where $B_n = B_X(n, k, d, a)$, and a is arbitrary.*

PROOF. By definition, if $v \in B_n(X)$ then $v = x_k^e w$ and $w \in M^{k,a}(X)$. Since v is d-clean, Lemma 4.6(c) says that $w = y_{i_1} \cdots y_{i_s}$ is $(d-1)$-clean as a word in $\{y_1, \cdots, y_{a(k-1)}\} = Y$. Thus $w \in A_Y(s, a(k-1), d-1) = A_s(Y)$. By (*), $|A_s(Y)| \leqq f(s) \leqq f(n)$, since $s \leqq n$ and f is assumed to be monotonic.

Clearly $|\{x_k^e\}| = 1$ is polynomially bounded. Since every word in $B_n(X)$ can be written as a product of a word $x_k^e \in \{x_k^m \mid m > 0\}$, and $w \in A_s(Y)$ and the two sets have polynomial bounds, by 4.4 $|B_n(X)|$ is polynomially bounded.

LEMMA 4.10. *Under the hypothesis* (*) *and* (**), B'_n *is polynomially bounded in* n, *where* $B'_n = B'_x(n, k, d, a)$, *and* a *is arbitrary.*

PROOF. Let $w = x_{i_1} \cdots x_{i_n} \in B'_x(n, k, d, a)$ and let $0 \leqq j \leqq n$ be as small as possible such that $i_j = k$. Let $w_1 = x_{i_1} \cdots x_{i_{j-1}}$ and $w_2 = x_{i_j} \cdots x_{i_n}$, formally allowing $w_1 = 1$ if $i_j = 1$ and $w_2 = 1$ if x_k does not occur in w. Clearly, $w_1 \in A_X(j-1, k-1, d)$ and $w_2 \in B_x(n-j-1, k, d, a)$ and $w = w_1 w_2$. Since $|A_X(n, k-1, d)|$ and $|B_x(n, k, d, a)|$ are polynomially bounded (by (**) and 4.9, respectively) the proof now follows by 4.4.

LEMMA 4.11. *If* $w \in A(n, k, d)$, *then* w *can be factored as* $w = w_0 w_1 \cdots w_d$ *where* $w_0 \in B'(-, k, d, d)$ *and* $w_1, \cdots, w_d \in B(-, k, d, d)$.

PROOF. First, we claim that if $w \in A_n$ then w has no more than d distinct exponents of x_k which are $\geqq d$. For, if not, $w = v_0 x_k^{a_1} v_1 x_k^{a_2} \cdots x_k^{a_{d+1}} v_{d+1}$ where each of $a_1, \cdots, a_{d+1} \geqq d$, each of $v_1, \cdots, v_d \neq 1$, and each of $v_1, \cdots, v_d$ fails to begin with x_k. So $x_k > v_1, \cdots, v_d$. Consider the subword of $w, x_k^d v_1 x_k^{a_2} \cdots v_d$ and the factorization given by $u_1 = x_k^d v_1 x_k^{a_2 - d + 1}$, $u_2 = x_k^{d-1} v_2 x_k^{a_3 - d + 2}$, $u_3 = x_k^{d-2} v_3 x_k^{a_4 - d + 3}, \cdots, u_{d-2} = x_k^3 v_{d-2} x_k^{a_{d-1} - 2}$, $u_{d-1} = x_k^2 v_{d-1} x_k^{a_d}$, $u_d = v_d$. Clearly $u_1 > \cdots > u_d$, which yields a d-bad subword. This is a contradiction, since d-bad implies not d-clean, and so the first claim is proven.

Now given $w \in A_n$, write $w = w_0 x_k^{a_1} w_1 \cdots x_k^{a_s} w_s$ where the $a_1, \cdots, a_s$ record all occurrences of x_k to powers $\geqq d$. So $s < d$. Also, each $x_k^{a_i} w_i \in B$, since w_i does not begin with x_k or contain x_k to a power $\geqq d$. So $w \in B'B^s$, and, since $1 \in B$, $w \in B'B^d$.

PROOF OF THEOREM 4.8. In order to prove the polynomial bound on $|A(n, k, d)|$ use a double induction on k and d. Under the induction hypothesis (*) and (**), Lemmas 4.10 and 4.11 give that $|B(n, k, d, d)|$ and $|B'(n, k, d, d)|$ are polynomially bounded. The theorem now follows from 4.5 and 4.12 with $B^{(1)} = B'$, $B^{(2)} = \cdots = B^{(d+1)} = B$, and $t = d + 1$.

COROLLARY 4.12. $\dim_F(W_n / W_n \cap Q)$ *is polynomially bounded in* n.

PROOF. Immediate from Theorem 4.8 and Lemma 4.3.

This result has an application to growth rate of PI-algebras. The relevant definitions are to be found in [12].

THEOREM 4.13. *If A is a finitely generated* PI-*algebra, then A is polynomially bounded.*

PROOF. Assume that A is generated by $\{a_1, \cdots, a_k\}$ and that A satisfies a monic PI of degree d. Let $f(n)$ be the growth function determined by $\{a_1, \cdots, a_k\}$. Let $A_n = \mathrm{span}_F\{a_{i_1} \cdots a_{i_n} \mid 1 \leqq i_1, \cdots, i_n \leqq k\}$. In order to show that $f(n)$ is bounded by a polynomial, it suffices to show that $\dim_F|A_n$ is bounded by a polynomial in n.

There is a unique F-homomorphism Ψ from W_n onto A_n given by $\Psi(x_{i_1} \cdots x_{i_n}) = a_{i_1} \cdots a_{i_n}$. Since the rank of $W_n / W_n \cap Q$ is bounded by a polynomial, Theorem 4.13 will be proven if we show that $W_n \cap Q \subset \ker \Psi$. But, this is immediate from the definition of Q; if $f(x_1, \cdots, x_k) \in Q$, f vanishes on all k-tuples from A, so $\Psi(f) = f(a_1, \cdots, a_k) = 0$.

COROLLARY 4.14. *If the cocharacter of Q is $\Sigma_{\lambda \in \mathrm{Par}(n)} m_\lambda \chi_\lambda$, then $\Sigma_{\Lambda_k(n)} m_\lambda$ is polynomially bounded in n.*

PROOF. If the GL(k) character ϕ_λ has degree h_λ then by Theorem 2.9, $\dim_F(W_n / W_n \cap Q) = \Sigma_{\Lambda_k(n)} m_\lambda h_\lambda \geqq \Sigma_{\Lambda_k(n)} m_\lambda$. Since the former is polynomially bounded, the latter must be.

COROLLARY 4.15. *If Q contains a Capelli polynomial, then $\Sigma_{\lambda \in \mathrm{Par}(n)} m_\lambda$ is polynomially bounded in n.*

PROOF. By [11], if Q contains a Capelli identity it contains all polynomials corresponding to idempotents on partitions of height greater than or equal to some fixed height. So, the cocharacter is contained in a strip and the result follows from Corollary 4.14.

REMARKS. (1) Since the methods of this section are constructive, explicit polynomial bounds can be given in Theorems 4.8, and 4.12–4.15. We don't do this, as the methods do not seem to be efficient. It should be noted, however, that all of these polynomials have degrees which are bounded by functions of k and d.

(2) The converse of Theorem 4.13 is not true. For a counterexample let F be a field of characteristic zero and let G be a nilpotent group. Then $F[G]$ will be polynomially bounded, ([2]), but it will not be a PI-algebra unless G is abelian by finite ([9], 3.7). It would be interesting to know if there is a setting in which a converse of 4.13 holds.

(3) Since $\Sigma_{\lambda \in \Lambda_k(n)} m_\lambda$ is polynomially bounded, it is reasonable to conjecture that $\Sigma_{\mathrm{Par}(n)} m_\lambda$ is also polynomially bounded.

(4) dim $W_n/W_n \cap Q$ will seldom be less than polynomial. More precisely, if Q does not contain X^m, then dim $W_n/W_n \cap Q$ will be bounded below by a polynomial of degree $k-1$ in n.

(5) Since Theorem 4.8 makes no reference to F, Corollary 4.12 and Theorem 4.13 will hold whether or not F has characteristic zero. Moreover, if F is any commutative ring with unit, 4.12 will hold with "dimension" replaced by "minimal number of generators" and 4.13 will hold for a suitable analogue of rate of growth.

5. Applications to matrices

In this last section we discuss the important special case in which $Q = \{f(x_1,\cdots,x_k) \mid f$ an identity for F_r, the $r \times r$ matrices$\}$. Then $R = F\langle x_1,\cdots,x_k\rangle/Q$ is a generic matrix algebra, i.e., R is isomorphic to $F[X_1,\cdots,X_k]$ where $X_l = (y_{ij}^{(l)})_{i,j=1,\cdots,r}$, $l = 1,\cdots,k$ and where the $y_{ij}^{(l)}$ are commuting, independent indeterminants. $W_n/W_n \cap Q$ corresponds to the space of homogeneous polynomials in $X_1,\cdots,X_k$ of degree n. Moreover, if $f(n) = \dim W_n/W_n \cap Q$, then $g(n) = \Sigma_{i=1}^n f(i)$ is a growth function for R, since R is generated by $X_1,\cdots,X_k$. We now quote two relevant theorems, the first is due to Malliavin-Brameret [8] and the second to Procesi [5].

LEMMA 5.1. *The rate of growth of R is $[n^t]$ where t is the transcendence degree of the quotient field of the center of R.*

LEMMA 5.2. *The quotient field of the center of R has transcendence degree $r^2(k-1)+1$ over F.*

So we get as immediate consequences:

COROLLARY 5.3. *If m_λ = the multiplicity of λ in the (either homogeneous or multilinear, by 2.7) cocharacter of Q, and if $h(\lambda,k)$ = the degree of the irreducible* GL(k)*-character on λ, then $\Sigma_{i\le n}\Sigma_{\lambda\in\Lambda_k(i)} h(\lambda,k)m_\lambda$ is bounded above and below by polynomials in n of degree $r^2(k-1)+1$.*

PROOF. $f(i) = \dim W_i/W_i \cap Q = \Sigma_{\lambda\in\Lambda_k(n)} h(\lambda,k)m_\lambda$, and $g(n) = \Sigma_{i=1}^n f(i)$.

COROLLARY 5.4. *With notation as in 5.3, $f(n) = \Sigma_{\lambda\in\Lambda_k(n)} h(\lambda,k)m_\lambda$ is bounded above and below by polynomials in n of degree $r^2(k-1)$.*

PROOF. First, we claim that f is a non-decreasing function. In order to show this it suffices to show that there is an injection $W_n/W_n \cap Q \to W_{n+1}/W_{n+1} \cap Q$. Let $f(x_1,\cdots,x_k) \in W_n$, then $x_1 f(x_1,\cdots,x_k) \in W_{n+1}$, and $x_1 f(x_1,\cdots,x_k) \in Q$ if and only if $f(x_1,\cdots,x_k)\in Q$. This is by a well known theorem of Amitsur which states that if $f\cdot g\in Q$ then either $f\in Q$ or $g\in Q$.

Now, assume by 5.3 that $c_1 n^{r^2(k-1)+1}\leqq g(n)\leqq c_2 n^{r^2(k-1)+1}$. Then $c_1 n^{r^2(r-1)+1}\leqq g(n)=\Sigma_{i=1}^n f(i)\leqq \Sigma_{i=1}^n f(n)=nf(n)$. So $f(n)\geqq c_1 n^{r^2(k-1)}$. On the other hand, $c_2(2n)^{r^2(k-1)+1}\geqq g(2n)=\Sigma_{i=1}^{2n} f(i)\geqq \Sigma_{i=n+1}^{2n} f(i)\geqq \Sigma_{i=n+1}^{2n} f(n)=nf(n)$. So $f(n)\geqq 2^{r^2(r-1)+1}\cdot c_2 n^{r^2(k-1)}$

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